Maps between curves

Recall from the exercises that a map $X \to Y$ between two ns proper curves is finite & surjective or constant.

Setting: X, Y ns connected curves / $k = \bar{k}$
$\pi : X \to Y$ finite map

Def $\deg(\pi) := [K(X)/K(Y)]$ degree of π

ramification index at $p \in |X|$

$$e_p := \mathrm{ord}_p(\pi^b(t)) \quad \text{for } t \in \mathcal{O}_{Y,\pi(p)} \text{ local parameter}$$

if $e_p > 1$ say π is ramified at p and $\pi(p)$ is a branch pt

$\begin{cases} \text{if } \mathrm{char}(k) \mid e_p & \text{wildly ramified} \\ \text{if } \mathrm{char}(k) \nmid e_p & \text{tamely ramified} \end{cases}$

pullback of divisors $\pi^*: \mathrm{Div}(Y) \to \mathrm{Div}(X)$

defined by $\pi^*(q) = \sum_{p \in |\pi^{-1}(q)|} e_p \, p$

Lemma $\pi^* \mathcal{O}_Y(D) \cong \mathcal{O}_Y(\pi^* D)$

Pf Suffices to check $\mathrm{ord}_p(\pi^b(g)) = e_p \, \mathrm{ord}_q(\pi^b(g))$ for all $g \in K(Y)$

write $g = a\, t^{\mathrm{ord}_q(g)}$

$\Rightarrow \mathrm{ord}_p(\pi^b(g)) = \mathrm{ord}_p(t^{\mathrm{ord}_q(g)}) = \mathrm{ord}_q(g) \cdot e_p$ $\square$

Want to compare genus of X and Y

Def $\pi : X \to Y$ is seperable if $K(X)/K(Y)$ is seperable.

Lemma L/K seperable field extension
$\Rightarrow \Omega_{L/K} = 0$

Proof Let $a \in L$, $f \in K[t]$ s.t. $f(a) = 0$ and $\frac{\partial f}{\partial t}(a) \neq 0$

Then $0 = d(f(a)) = \frac{\partial f}{\partial t}(a)\, da$

$\frac{\partial f}{\partial t}(a) \neq 0 \Rightarrow da = 0$ $\square$

Prop $\pi : X \to Y$ seperable.

Then $0 \to \pi^* \Omega_Y \to \Omega_X \to \Omega_{X/Y} \to 0$ is a SES. $(*)$

Proof exactness of $\pi^*\Omega_Y \to \Omega_X \to \Omega_{X/Y} \to 0$ is true in general.

show $\pi^*\Omega_Y \to \Omega_X$ is injective.

Lemma $\Rightarrow (\Omega_{X/Y})_{\eta_X} = \Omega_{K(X)/K(Y)} = 0$.

$\Rightarrow (\pi^*\Omega_Y)_{\eta_X} \to (\Omega_X)_{\eta_X}$ iso

$\Rightarrow \pi^*\Omega_Y \to \Omega_X$ $\square$

Rmk The Proposition and its proof works more generally: $\pi : X \to Y$ dominant s.t. $K(X)/K(Y)$ finite seperable.

Proposition $\pi : X \to Y$ finite seperable

(a) $\Omega_{X/Y}$ is a torsion sheaf and

$$\mathrm{supp}(\Omega_{X/Y}) = \{ p \in |X| \mid \pi \text{ ramified at } p \}$$

(b) for all $p \in |X|$ $(\Omega_{X/Y})_p$ is a principal $\mathcal{O}_{X,p}$ module of finite length

$$\mathrm{len}_{\mathcal{O}_{X,p}} (\Omega_{X/Y})_p = \mathrm{ord}_p \left(\frac{\partial \pi^b(s)}{\partial s} \right)$$

where $t \in \mathcal{O}_{Y,\pi(p)}$, $s \in \mathcal{O}_{X,p}$ are local parameters.

(c) π tamely ramified at $p \Rightarrow \mathrm{len}_{\mathcal{O}_{X,p}} (\Omega_{X/Y})_p = e_p - 1$

π wildly ramified at $p \Rightarrow \mathrm{len}_{\mathcal{O}_{X,p}} (\Omega_{X/Y})_p > e_p - 1$

Proof local calculation

SES $(*)$ $\Rightarrow$ $\Omega_{X/Y}$ torsion

$(\Omega_{X/Y})_p = 0$ iff $\pi^b(dt) \in \Omega_{X,p} = \mathcal{O}_{X,p}\, ds$

$$\pi^b(dt) = d(\pi^b(t)) = d(u s^{e_p}) = s^{e_p} \underbrace{du}_{=0} + u e_p s^{e_p - 1} ds$$

$$(*)_p: \quad 0 \to \mathcal{O}_{X,p}\,\pi^\flat(dt) \to \mathcal{O}_{X,p}\,ds \to (\Omega_{X/Y})_p \to 0$$

$$\Rightarrow (\Omega_{X/Y})_p \cong \mathcal{O}_{X,p}/(e_p s^{e_p-1})$$

Now can interpret everything in the proposition. $\square$

<u>Def</u> $\pi: X \to Y$ finite separable

<u>ramification divisor</u> $\mathrm{Ram}(\pi) = \sum_{p \in |X|} \mathrm{len}_{\mathcal{O}_{X,p}} (\Omega_{X/Y})_p \, p \in \mathrm{Div}(X)$

<u>Rmk</u> $\mathrm{Ram}(\pi)$ is effective and can interpret it as a finite subscheme $\bigsqcup_{p \in \mathrm{supp}(\pi)} \mathrm{Spec}(\mathcal{O}_{X,p}/(e_p s^{e_p-1}))$

<u>Theorem</u> (Riemann – Hurwitz)
$\pi: X \to Y$ finite separable

Then $\Omega_X \cong \pi^* \Omega_Y \otimes \mathcal{O}_X(\mathrm{Ram}(\pi))$
$K_X \sim K_Y + \mathrm{Ram}(\pi)$

In particular,

$$2g(X)-2 = \deg(\pi)(2g(Y)-2) + \deg(\mathrm{Ram}(\pi))$$

$$= \deg(\pi)(2g(Y)-2) + \sum_{p \in |X|} (e_p - 1) \quad \text{if } \pi \text{ tamely ramified.}$$

<u>Proof</u> $(*) \otimes \Omega_X^\vee: \quad 0 \to \pi^*\Omega_Y \otimes \Omega_X^\vee \to \mathcal{O}_X \to \mathcal{O}_{\mathrm{Ram}(\pi)} \to 0$

$$\Rightarrow \pi^*\Omega_Y \otimes \Omega_X^\vee = \mathcal{O}_X(-\mathrm{Ram}(\pi)) \quad \text{ideal sheaf} \qquad \square$$

<u>Example</u>: $Y = \mathbb{P}^1 \quad \pi: X \xrightarrow{d:1} \mathbb{P}^1 \quad \deg(\pi) = d$

$$\text{RH} \Rightarrow \quad 2g(X) - 2 = -2d + \deg(\mathrm{Ram}(f))$$

$$\Rightarrow \deg(\mathrm{Ram}(f)) = 2g(X) + 2d - 2$$

Def Suppose $\mathrm{char}(k) = p$. $\quad \mathrm{Frob} : k \to k \ ; \quad a \mapsto a^p$
S scheme over k

$$\begin{tikzcd} S_\Phi \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dr, bend right=60, "\text{structure morphism}"'] & S \arrow[d] \\ \mathrm{Spec}(k) \arrow[r, "\mathrm{Frob}"] & \mathrm{Spec}(k) \end{tikzcd}$$

$\Phi = \Phi_S : S_\Phi \longrightarrow S$ <u>geometric Frobenius</u> of S

Example $\Phi : Y_\Phi \to Y$ is finite of degree p:

$$K(Y_\Phi) \cong k \otimes_{\mathrm{Frob}, k} K(Y) \cong \{ a \in \overline{K(Y)} \mid a^p \in K(Y) \} = K(Y)^{1/p}$$

$y \in |Y_\Phi| \quad \Phi^\flat : \mathcal{O}_{Y, \Phi(y)} \longrightarrow \mathcal{O}_{Y_\Phi, y} \qquad a \mapsto a^p$

$\qquad t \longmapsto u s^p \qquad$ "$s = t^{1/p}$"

$\qquad u \in k^\times$

$\Rightarrow \Phi$ wildly ramified everywhere.

Def $\pi : X \to Y$ is <u>purely inseparable</u> if $K(X)/K(Y)$ is purely inseparable

Prop $\pi : X \to Y$ purely inseparable

Then $X \cong Y$ as abstract schemes (i.e. over $\mathrm{Spec}(\mathbb{Z})$) and π is a composition of geometric Frobenius morphisms
In particular, $g(X) = g(Y)$

Proof (sketch) π purely inseparable $\Rightarrow \deg(\pi) = p^r$

$\Rightarrow K(X)^{p^r} \subseteq K(Y) \qquad$ (i.e. $\forall a \in K(X) \quad a^{p^r} \in K(Y)$)

$\Leftrightarrow K(X) \subseteq K(Y)^{1/p^r} = K(Y_{\Phi^r})$

$[K(X)/K(Y)] = p^r = [K(Y_{\Phi^r})/K(Y)]$

$\Rightarrow K(X) \cong K(Y_{\Phi^r}) \quad \Rightarrow \quad X \cong Y_{\Phi^r}$ over $\mathrm{Spec}(\mathbb{Z})$

equivalence of categories $\quad \{\text{ns proper curves w/ dominant maps}\} \leftrightarrow \{\text{fields of tr. deg } 1\}$ $\qquad \square$

Rmk Can factor any $\pi : X \to Y$ as

$$\begin{tikzcd} X \arrow[rr, "\pi"] \arrow[dr, "\text{purely insep.}"'] & & Y \\ & X' \arrow[ur, "\text{sep.}"'] & \end{tikzcd}$$

Canonical morphism

X ns proper curve $/k$

Lem if $g(X) \geq 1$, then ω_X is globally generated.

Pf $h^0(\omega_X(-p)) \overset{SD}{\cong} h^1(\mathcal{O}_X(p)) = g(X) - 1 + \underbrace{\deg(p)}_{=1} - \underbrace{h^0(\mathcal{O}_X(p))}_{=1}$

$= g(X) - 1.$ by. ExSheet 1. $\square$.

Def $g(X) \geq 1$ $\varphi: X \to \mathbb{P}(H^0(\omega_X))$
canonical morphism (unique up to $\mathrm{Aut}_k(\mathbb{P}(H^0(\omega_X))$)

Hyperelliptic curves:

Def A ns proper curve X is hyperelliptic if there is a deg 2 morphism $\pi: X \xrightarrow{2:1} \mathbb{P}^1$

Example $g(X) = 2$ ω_X globally generated: $\pi: X \to \mathbb{P}(H^0(\omega_X))$

$$\deg(\pi) = \deg(\pi^* \mathcal{O}(1)) = \deg(\pi^* \mathcal{O}(1)) = \deg(\omega_X) = 2.$$

Prop $g(X) \geq 1$ TFAE

(1) X is hyperelliptic

(2) There is a line bundle $\mathcal{L}$ on X with $\deg(\mathcal{L}) = h^0(\mathcal{L}) = 2$

(3) the canonical morphism of X is not an embedding. (ω_X is not very ample)

Proof (1) $\Rightarrow$ (2): $\pi: X \xrightarrow{2:1} \mathbb{P}^1$ hyperelliptic cover.
Show that $\mathcal{L} := \pi^* \mathcal{O}(1)$ is the desired line bundle.
$\deg(\mathcal{L}) = \deg(\pi) \deg(\mathcal{O}(1)) = 2.$ ✓

def of $\mathcal{L}$ $\Rightarrow$ $h^0(\mathcal{L}) \geq 2$

Suppose $h^0(\mathcal{L}) \geq 3$, then $h^0(\mathcal{L}(-p)) = 3$ or 2 $\forall p \in |X|$

if $h^0(\mathcal{L}(-p)) = 3$, then $h^0(\underbrace{\mathcal{L}(-p-q)}_{\deg 0}) = 3$ or $2 > 1$ ↯

so $h^0(\mathcal{L}(-p)) = 2$ for all p.

$\Rightarrow h^0(\underbrace{\mathcal{L}(-p-q)}_{\deg 0}) = 2, 1$ $\Rightarrow$ $h^0(\mathcal{L}(-p-q)) = 1$ $\forall p, q \in |X|$

$\Rightarrow \pi$ embedding ↯

can more generally show that any $\mathcal{L}$ w/ $\deg(\mathcal{L}) \leq H^0(\mathcal{L})$ is globally generated.

$\Rightarrow \quad h^0(\mathcal{L}) \leq 2$

(2) $\Rightarrow$ (1) : Similar arguments $\Rightarrow h^0(\mathcal{L}) - h^0(\mathcal{L}(-p)) = 1 \quad \forall p \in |X|$
$\rightsquigarrow \quad X \to \mathbb{P}(H^0(\mathcal{L}))$ desired cover.

(2) $\Leftrightarrow$ (3) : $\varphi : X \longrightarrow \mathbb{P}(H^0(\omega_X))$ is an embedding iff

$$\begin{aligned} g-2 &= h^0(\omega_X(-p-q)) && \text{for all } p,q \in |X| \\ &= h^1(\mathcal{O}_X(p+q)) \\ &= g+1 - h^0(\mathcal{O}_X(p+q)) && (\text{Riemann-Roch}) \end{aligned}$$

$\Rightarrow \varphi$ embedding iff $\quad 2 = h^0(\mathcal{O}_X(p+q)) - 1 \quad \forall p,q \in |X|$

If $\mathcal{L}$ w/ $H^0(\mathcal{L}) = 2 = \deg(\mathcal{L})$, then $\mathcal{L} = \mathcal{O}_X(p_0 + q_0)$

$\Rightarrow \varphi$ not embedding.

If φ not embedding, $\exists\, p_0, q_0$ s.t. $h^0(\mathcal{O}_X(p_0 + q_0)) = 2$

$\Rightarrow \mathcal{O}_X(p_0 + q_0)$ is as in (2) $\quad \square$.

Do non-hyperelliptic curves exist?
If X $g(X) = 3$ nonhyperelliptic, $\deg(\omega_X) = 4 \Rightarrow X \subset \mathbb{P}(H^0(\omega_X))$
quartic. $\cong \mathbb{P}^2$

Example $\quad X_4 \overset{i}{\subset} \mathbb{P}^2$ quartic curve in $\mathbb{P}^2$

$$\Rightarrow g(X_4) = \frac{(4-1)(4-2)}{2} = 3$$

SES $\quad 0 \to i^*\mathcal{O}_{\mathbb{P}^2}(-4) \to i^*\Omega_{\mathbb{P}^2} \to \Omega_{X_4} \to 0$

$$\Rightarrow \underbrace{i^*\omega_{\mathbb{P}^2}}_{= \mathcal{O}_{\mathbb{P}^2}(-3)} = \omega_{X_4} \otimes i^*\mathcal{O}_{\mathbb{P}^2}(-4) \Rightarrow \omega_{X_4} = i^*_{\mathbb{P}^2}\, \mathcal{O}_{\mathbb{P}^2}(1)$$

$\Rightarrow$ canonical morphism of X_4 is an embedding.